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SOME PROBLEMS IN THE THEORY OF DIFFERENTIAL PURSUIT GAMES IN SYSTEMS WITH DISTRIBUTED PARAMETERS†

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A game with dynamics described by partial differential equations is considered. The equations of the players are additively represented on the right-hand side and are subject to integral or pointwise restrictions. The goal of the first player, who is informed of the instantaneous value of the control of his partner, is to bring the system into an unperturbed state. To solve the problem the decomposition method developed in [1] for a controlled system (with one player) is used. Three combinations of restrictions on the players are considered. In all cases the control of the first player is presented explicitly. The main complication, compared with the problem considered previously [1] is that this control consists of two terms estimated in different norms.

Suppose that in the space $L_2(\Omega)$ a differential operator A is specified of the form [2]

$$Az = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left((a_{ij}(x)\frac{\partial z}{\partial x_j}\right), \quad x \in \Omega \tag{1}$$

$$a_{ij}(x) = a_{ji}(x) \in C^1(\overline{\Omega})$$

where Ω is a bounded domain in R^n. The domain of definition $D(A)$ of the operator A is $\dot{C}^2(\Omega)$ (the space of doubly continuously differentiable finite functions). The coefficients $a_{ij}(x)$ satisfy the following condition: a constant $\gamma \neq 0$ exists such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant \gamma^2 \sum_{i=1}^{n} \xi_i^2, \quad \forall(\xi_1,\ldots,\xi_n) \in R^n, \quad x \in \Omega \tag{2}$$

which signifies the ellipticity of the operator A.

Putting

$$(z, y)_A = (Ax, y), \quad z, y \in \dot{C}^2(\Omega)$$

it can be shown that $(\,.\,,.\,)_A$ satisfies all the requirements of a scalar product. Thus $\dot{C}^2(\Omega)$ is turned into a Hilbert space. But it is incomplete with respect to the norm generated by the scalar product $(\,.\,,.\,)_A$. Completing the space $\dot{C}^2(\Omega)$ relative to the norm

$$\|z\|_A = \sqrt{(Az, z)}, \quad z \in \dot{C}^2(\Omega)$$

we obtain a complete Hilbert space, called the energy space of the operator A and denoted by H_A.

An operator A with condition (2) has a discrete spectrum [3], i.e. it has an infinite sequence $\lambda_1, \lambda_2, \ldots$ of generalized numbers with a unique limit at infinity and a sequence $\varphi_1, \varphi_2, \ldots$ of generalized eigenelements, complete in the space $L_2(\Omega)$. Without loss of generality, we can put $(\varphi_i, \varphi_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

Using these facts we can construct the following spaces (which are, of course, associated with the operator A) [4]. Let

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$$l_r = \left\{ \alpha = (\alpha_1, \alpha_2, \ldots): \sum_{i=1}^{\infty} \lambda_i^r \alpha_i^2 < \infty \right\}$$

$$H_r(\Omega) = \left\{ f \in L_2(\Omega): f = \sum_{i=1}^{\infty} \alpha_i \varphi_i, \ \alpha \in l_r \right\}$$

In the spaces l_r, $H_r(\Omega)$ we define scalar products as follows:

$$(\alpha, \beta)_r = \sum_{i=1}^{\infty} \lambda_i^r \alpha_i \beta_i, \quad \alpha, \beta \in l_r$$

From this

$$\|\alpha\| = \left(\sum_{i=1}^{\infty} \lambda_i^r \alpha_i^2 \right)^{1/2}$$

Similarly

$$(f, g)_r = (\alpha, \beta)_r; \quad f = \sum_{i=1}^{\infty} \alpha_i \varphi_i, \quad g = \sum_{i=1}^{\infty} \beta_i \varphi_i$$

From this $\| f \| = \| \alpha \|$.

We note that $H_0(\Omega) = L_2(\Omega)$ and $H_r(\Omega) \subset H_s(\Omega)$ when $s < r$.

We will denote by $C(0, T; H_r(\Omega))(L_2(0, T; H_r(\Omega)))$ the space consisting of continuous (measurable) functions on $[0, T]$ with values in $H_r(\Omega)$, where T is a positive constant.

The equations encountered below are to be interpreted using the theory of distributions (generalized functions) [4].

We consider the following differential game

$$\begin{aligned} & dz(t)/dt + Az(t) = -u(t) + \upsilon(t), \quad 0 < t \leq T \\ & u(\cdot), \upsilon(\cdot) \in L_2(0, T; \ H_r(\Omega)) \\ & z(0) = z^0, \quad z^0 \in H_{r+1}(\Omega) \end{aligned} \tag{3}$$

The operator A is specified in the form (1).

It was proved in [4] that a unique solution of problem (3) exists in the space $C(0), T; H_{r+1}(\Omega))$ if $z^0 \in H_{r+1}(\Omega)$ for some $r \geq 0$.

The functions $u(t)$, $\upsilon(t)$, $0 \leq t \leq T$ are called controls of the first (pursuing) and second (evading) player, respectively. They are constricted by restrictions given by one of the following systems of inequalities

$$\|u(\cdot)\| \leq \rho, \quad \|\upsilon(\cdot)\| \leq \sigma \tag{4}$$

$$\|u(\cdot)\| \leq \rho, \quad \|\upsilon(t)\| \leq \sigma, \quad 0 \leq t \leq T \tag{5}$$

$$\|u(t)\| \leq \rho, \quad 0 \leq t \leq T, \quad \|\upsilon(\cdot)\| \leq \sigma \tag{6}$$

where ρ, and σ are non-negative constants.

Controls $u(t)$, $\upsilon(t)$, $0 \leq t \leq T$ which satisfy one of the conditions (4)–(6) are called admissible.

Definition. We say that in the game (3), (4) or (3), (5) or (3), (6) one can complete the pursuit from the initial point z^0 if a number $T = T(z^0) \geq 0$ exists such that for any admissible control $\upsilon(\cdot)$ of the evading player, knowing at every time $t \in [0, T]$ the equation from (3) and the value of $\upsilon(t)$, one can choose the value of $u(t)$ such that $u(\cdot)$ is an admissible control of the pursuing player and $z(t') = 0$ for the game (3), (4) and (3), (5), and $\sup_k |z_k(t')| \leq l$ for the game (3), (6), where $z_k(\cdot)$ is the Fourier coefficient, $l > 0$ is a constant, $t' \in [0, T]$, and $z(\cdot)$ is the solution of the corresponding problem with controls u, υ.

The pursuit problem. In game (3), (4) or (3), (5) or (3), (6) it is required to find, for every initial point

z^0 a guaranteed time $T(z^0)$ for finishing the game and to construct an admissible control $u(\cdot)$ for the pursuing player which satisfies the conditions of the definition given above.

Theorem 1. If

$$\rho > \sigma \tag{7}$$

then the pursuit problem is solvable for game (3), (4). Here

$$T(z^0) = \|z^0\|^2 / (\rho - \sigma)^2 \tag{8}$$

2. If

$$\rho^2 \geqslant 4\sigma \|z^0\| \tag{9}$$

then the pursuit problem is solvable for game (3), (5). Here

$$T(z^0) = [(\rho - \sqrt{\rho^2 - 4\sigma\|z^0\|}) / (2\sigma)]^2 \tag{10}$$

3. If $\rho > 0$, then the pursuit problem is solvable for game (3), (6).

Proof. 1. Using condition (7), the control $u(t)$, $0 \leqslant t \leqslant T$ of the pursuing player can be represented in the form [5]

$$u(t) = \upsilon(t) + w(t), \quad 0 \leqslant t \leqslant T \tag{11}$$

where $w(t)$, $0 \leqslant t \leqslant T$ is some as yet undefined function satisfying the inequality

$$\| w(\cdot) \| \leqslant \rho - \sigma \tag{12}$$

We will represent the solution $z(t)$ and the function $w(t)$ as Fourier series

$$z(t) = \sum_{k=1}^{\infty} z_k(t)\varphi_k, \quad w(t) = \sum_{k=1}^{\infty} w_k(t)\varphi_k, \quad z_k(\cdot), w_k(\cdot) \in L_2 \tag{13}$$

$$\sum_{k=1}^{\infty} \lambda_k^{r+1} \int_0^T |z_k(t)|^2 \, dt < \infty, \quad \sum_{k=1}^{\infty} \lambda_k^r \int_0^T |w_k(t)|^2 \, dt = \|w(\cdot)\|^2$$

Substituting expansions (13) into Eq. (3), using relation (11) and equating corresponding coefficients of the complete system $\{\varphi_k\}$, we obtain an infinite system of differential equations.

Integrating each equation of this system with appropriate initial conditions, we obtain

$$z_k(t) = e^{-\lambda_k t}\left(z_k^0 - \int_0^t e^{\lambda_k s} w_k(s) ds \right), \quad k = 1, 2, \ldots$$

where $z_k^0 = (z^0, \varphi_k)$ $(k = 1, 2, \ldots)$ are the Fourier coefficients of the function z^0.

It can be directly verified that when

$$w_k(t) = W_k \operatorname{sign} z_k^0, \quad 0 \leqslant t \leqslant T_0 \tag{14}$$

$$W_k = \lambda_k |z_k^0| / (e^{\lambda_k T_0} - 1), \quad T_0 = T(z^0) = \|z^0\|^2 / (\rho - \sigma)^2$$

we have the equalities

$$z_k(T_0) = 0, \quad k = 1, 2, \ldots \tag{15}$$

and the function $w(t)$, $0 \leqslant t \leqslant T$ satisfies inequality (12) when $T = T_0$. From this we have the admissibility of the control (11)

$$\|u(\cdot)\| \leqslant \|\upsilon(\cdot)\| + \|w(\cdot)\| \leqslant \sigma + \rho - \sigma = \rho.$$

Because all the Fourier coefficients $z_k(t)$, $0 \leqslant t \leqslant T$ vanish at $t = T_0$ by virtue of (15), we have $z(T_0) = 0$.

2. We will choose the control $u(t)$, $0 \leqslant t \leqslant T$ of the pursuing player in the form (11). The function $w(t)$, $0 \leqslant t \leqslant T$ is given by the second equation in (13) and formula (14). If $T(z^0)$ is given by (10), then $\| w(\cdot) \|^2 \leqslant T^{-1} \| z^0 \|^2$, i.e. $w(\cdot) \in L_2(0, T; H_r(\Omega))$.

From this we have $\| w(\cdot) \| \leqslant \rho - \sigma\sqrt{T}$. Noting that $\| \upsilon(\cdot) \| \leqslant \sigma\sqrt{T}$, we find that $\| u(\cdot) \| \leqslant \rho$, i.e. the control ((12)) is admissible.

It can be shown by direct computation that $z_k(T) = 0$ $(k = 1, 2, \ldots)$, i.e. $z(T) = 0$.

3. Because the pursuer knows the value of the evading player's control $\upsilon(t)$ at every time t, the pursuer can represent $\upsilon(t)$ in the form $\upsilon_1(t) + \upsilon_2(t)$, $0 \leqslant t \leqslant T$, where the function $\upsilon_1(\cdot)$, $\upsilon_2(\cdot)$ is constructed as follows:

$$\upsilon_1(t) = \begin{cases} \upsilon(t), & \text{if} \quad \|\upsilon(t)\| \leqslant \bar{\rho} \\ \bar{\rho}\upsilon(t)/\|\upsilon(t)\|, & \text{if} \quad \|\upsilon(t)\| > \bar{\rho} \end{cases}$$

$$\upsilon_2(t) = \begin{cases} 0, & \text{if} \quad \|\upsilon(t)\| \leqslant \bar{\rho} \\ (\|\upsilon(t)\| - \bar{\rho})\upsilon(t)/\|\upsilon(t)\|, & \text{if} \quad \|\upsilon(t)\| > \bar{\rho} \end{cases}$$

where $\bar{\rho}$ is some fixed non-negative constant satisfying the inequality $\bar{\rho} > \rho$.

If one chooses a control $u(t)$, $0 \leqslant t \leqslant T$ in the form (11), where $\| w(t) \| \leqslant \rho - \bar{\rho}$, $0 \leqslant t \leqslant T$, then the right-hand side of the equation consists of two controls which direct the first and second players, respectively. It has been shown [1] that a time T exists such that using the control $w(\cdot)$ one can take the trajectory to the origin of coordinates. Hence the second player, controlling the function $\upsilon(\cdot)$, strives to violate the inequality $\sup_k | z_k(T) | \leqslant l$.

Thus, to solve the corresponding differential equations we have

$$\begin{gathered} z_k(T) = -e^{-\lambda_k T} \int_0^T e^{\lambda_k t} \upsilon_{2k}(t)dt \\ \upsilon_{2k}(t) = (\upsilon_2(t), \varphi_k), \quad 0 \leqslant t \leqslant T, \quad k = 1, 2, \ldots \end{gathered} \tag{16}$$

We suppose that $\| \upsilon(\cdot) \| \leqslant \sigma_1$ where $0 < \sigma_1 \leqslant \sigma$. Then for fixed $k(\geqslant 1)$ we have

$$|\upsilon_{2k}(t)| \leqslant (\|\upsilon(t)\| - \bar{\rho})\lambda_k^{-r/2}$$

where equality is achieved if $\upsilon(t) = \upsilon_k(t)\varphi_k$, where $\upsilon_k(t) = (\upsilon(t), \varphi_k))$, $0 \leqslant t \leqslant T$. From this and from (16) we obtain

$$|z_k(T)| \leqslant I_k, \quad I_k = e^{-\lambda_k T} \int_0^T e^{\lambda_k t} \big|\|\upsilon(t)\| - \bar{\rho}\big| dt \lambda_k^{-r/2}$$

Let

$$\max_{\|\upsilon(\cdot)\| \leqslant \sigma_1} I_k \leqslant \left(\sigma_1 \left[\frac{1 - e^{2\lambda_k(\bar{T} - T)}}{2\lambda_k} \right]^{1/2} - \bar{\rho} \frac{1 - e^{\lambda_k(\bar{T} - T)}}{\lambda_k} \right) \lambda_k^{r/2} = \Psi(\bar{T}).$$

It can be shown that equality is achieved if

$$\upsilon(t) = \sigma_1 e^{\lambda_k t} \left[\left(\frac{e^{2\lambda_k T} - e^{2\lambda_k \bar{T}}}{2\lambda_k} \right)^{1/2} \varphi_k \right]^{-1}$$

$$0 \leqslant \bar{T} \leqslant t \leqslant T, \quad \|\upsilon(\cdot)\| \leqslant \sigma_1, \quad \upsilon(t) \geqslant \bar{\rho}, \quad \bar{T} \leqslant t \leqslant T$$

Using the methods of differential calculus one can verify that

$$\Psi_k = \max \Psi_k(\bar{T}) = \begin{cases} \Psi_k(0), & T \leqslant \Phi_k \\ \dfrac{\sigma_1^2}{\sqrt{2}((2\bar{\rho}^2 + \lambda_k \sigma_1^2)^{1/2} + \sqrt{2}\bar{\rho})\lambda_k^{r/2}}, & T > \Phi_k \end{cases}$$

$$\Phi_k = \ln(1 + \sigma_1^2 \lambda_k / (2\rho_k^2)) / (2\lambda_k)$$

Thus, if $\max_{k \geqslant 1} \Psi_k \leqslant l$, the game ends after a time T. Otherwise, i.e. if $|z_k(T)| > l$ for some $k(\geqslant 1)$, then $l < \Psi_k$ and from this we have $\sigma_1 \geqslant c = \text{const} > 0$.

Taking $z(T)$ as a new initial position, using similar arguments we arrive at the following conclusion: either the game ends after a time $T + T_1$, or the second player gain loses an amount of energy not less than c. Thus, after a finite number of steps the energy of the second player is exhausted, which means the end of the game.

Example. Suppose that heat is propagating along a rod of unit length whose ends are maintained at zero temperature. We then have problem (3) where $Az = -d^2z/dx^2$ and the domain of definition of A is the subspace $\dot{C}^2(0, 1)$ of the space $L_2(0, 1)$. The function z^0 has the form

$$z^0 = \frac{1}{48\sqrt{2}} \begin{cases} -4x^3 + 3x, & 0 \leqslant x \leqslant 1/2 \\ -4(1-x)^3 + 3(1-x), & 1/2 \leqslant x \leqslant 1 \end{cases}$$

It has been shown [2] that the energy space H of operator A is the space $\dot{W}_2^1(0, 1)$ consisting of functions satisfying the following conditions

1. they are absolutely continuous on [0, 1];
2. their first derivatives are square integrable on [0, 1];
3. they vanish at the points $x = 0, x = 1$.

The generalized eigenfunctions and eigenvalues of the operator A have the form [2]

$$\varphi_k = \sqrt{2} \sin \pi k x, \quad 0 \leqslant x \leqslant 1, \quad \lambda_k = (\pi k)^2, \quad k = 1, 2, \ldots$$

Using this information we construct the space $H_r(0, 1)$, $r \geqslant 0$.

Case 1. Suppose the constraints $\| u(\cdot) \| \leqslant 2$, $\| \upsilon(\cdot) \| \leqslant 1$ are imposed on the control functions $u(t)$, $\upsilon(t)$, $0 \leqslant t \leqslant T$.

One can verify that $z^0 \in H_r(0, 1)$ when $r = 3$. Hence if $u(\cdot), \upsilon(\cdot) \in H_2(0, 1)$ then problem (3) has a unique solution $z(\cdot)$ in the space $C(0, T; H_3(0, 1))$. Applying Section 1 of the theorem we find that the pursuit can be terminated after a time

$$T = \frac{1}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{1}{8}$$

The control $u(t)$, $0 \leqslant t \leqslant T$ for the pursuing player has the form (11) where

$$w(t) = \sum_{k=1}^{\infty} \frac{\sqrt{2}(-1)^{(k+3)/2}(1 + (-1)^{k+3})}{2(\pi k)^2 (e^{\lambda_k} - 1)} \sin \pi k x$$

Case 2. Suppose the constraints $\| u(\cdot) \| \leqslant 2$, $\| \upsilon(t) \| \leqslant 1$, $0 \leqslant t \leqslant T$ are imposed on the control functions $u(\cdot)$, $\upsilon(\cdot)$. Because $\| z^0 \| = 1/(2\sqrt{2})$, inequality (9) is satisfied. Hence we apply Section 2 of the theorem. The completion time of the game is $T = 2 - (\sqrt{(2)}/4 + \sqrt{(4)} - \sqrt{(2)})$.

Case 3. We consider the preceding example with constraints $\| u(t) \| \leqslant \rho$, $0 \leqslant t \leqslant T$, $\| \upsilon(\cdot) \| \leqslant \sigma$.

We choose the control $u(t)$, $0 \leqslant t \leqslant T$ of the pursuing player in the form (11) where $\| w(t) \| \leqslant \rho - \bar{\rho}$, $0 \leqslant t \leqslant T$ or

$$\sum_{k=1}^{\infty} \lambda_k^3 w_k(t) \leqslant (\rho - \bar{\rho})^2$$

As in [1] we have

$$T = \| z^0 \| / (\rho - \bar{\rho}) = 1 / [2\sqrt{2}(\rho - \bar{\rho})] \tag{17}$$

for the completion time of the game.

If we put $\rho = 2$, $\bar{\rho} = 1$, $\sigma = 1$, it can be shown

$$\Psi_k = \sigma_1^2 / (\sqrt{2}(\sqrt{2+\lambda_k\sigma_1^2} + \sqrt{2})\lambda_k^{3/2})$$

where $0 < \sigma_1 \leqslant 1$.

From this it follows that

$$\max_k \Psi_k = \zeta\sigma_1^2, \quad \zeta = ((2+\sqrt{4+\sqrt{2}\pi^2\sigma_1^2})\pi^3)^{-1}$$

This means that if

$$1/((2+\sqrt{4+\sqrt{2}\pi^2})\pi^3) \leqslant l$$

then one can complete the pursuit in a time T (17). Otherwise one must transfer to the next step. Here we note that the evading player must lose some of his energy. The game thus concludes after a finite number of steps.

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